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Filtered-graded transfer of noncommutative Gröbner bases

Transferencia filtrado-graduado de bases de Gröbner no conmutativas

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Abstract. As the case of free $\mathbb{k}$-algebras and PBW algebras, given a bijective skew PBW extension A, we will show that it is possible transfer Gröbner bases between A and its associated graded ring.

Key words and phrases. Noncommutative Gröbner bases; skew PBW extensions; filtered module; graded module.

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Resumen. Como en el caso de $\mathbb{k}$-álgebras libres y PBW álgebras, dada A una extensión PBW torcida biyectiva, mostraremos que es posible transferir bases de Gröbner entre A y su anillo graduado asociado.

Palabras y frases clave. Bases de Gröbner no conmutativas; extensiones PBW torcidas; módulo filtrado; módulo graduado.

1. Introduction

In [8] it was shown that if $A = \mathbb{k}[a_i]_{i\in\Lambda}$ is a $\mathbb{k}$-algebra generated by $\{a_i\}_{i\in\Lambda}$ over the field $\mathbb{k}$, and I a left ideal of A, then a nonempty subset G of I is a Gröbner basis for I if, and only if, $\overline{G}$ is a Gröbner basis of $Gr(I)$, where $\overline{G}$ denotes the image of G in $Gr(A)$ and $Gr(I)$ is the left ideal associated to I in $Gr(A)$. A similar fact is proved in [1] for the case of PBW algebras. We will present an analogous result for skew PBW extensions, another class of noncommutative rings and algebras of polynomial type that generalize classical PBW extensions and include many important types of quantum algebras.

2. Skew PBW extensions

In this section we recall the definition of skew PBW (Poincaré-Birkhoff-Witt) extensions defined firstly in [4], and we will review also some basic properties about the polynomial interpretation of this kind of noncommutative rings. Two particular subclasses of these extensions are recalled also.

Definition 2.1. Let R and A be rings. We say that A is a *skew PBW extension of R* (also called a $\sigma - PBW$ extension of R) if the following conditions hold:

(i) $R \subseteq A$.

(ii) There exist finite elements $x_1, \dots, x_n \in A$ such A is a left R-free module with basis

$$\mathrm{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

(iii) For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R. \tag{1}$$

(iv) For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n. \tag{2}$$

Under these conditions we will write $A := \sigma(R)\langle x_1, \dots, x_n\rangle$.

A particular case of skew PBW extension is when all derivations δ_i are zero. Another interesting case is when all σ_i are bijective and the constants c_{ij} are invertible. We recall the following definition (cf. [4]).

Definition 2.2. Let A be a skew PBW extension.

(a) A is quasi-commutative if the conditions (iii) and (iv) in Definition 2.1 are replaced by

(iii') For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that

$$x_i r = c_{i,r} x_i. \tag{3}$$

(iv') For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that

$$x_j x_i = c_{i,j} x_i x_j. \tag{4}$$

(b) A is bijective if σ_i is bijective for every $1 \leq i \leq n$ and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

The skew PBW extensions can be characterized in a similar way as it was done in [1] for PBW rings (see Proposition 2.4 there in).

Theorem 2.3. *Let A be a left polynomial ring over R w.r.t.. $\{x_1, \dots, x_n\}$. A is a skew PBW extension of R if and only if the following conditions hold:*

(a) *For every $x^\alpha \in Mon(A)$ and every $0 \neq r \in R$ there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R - \{0\}$ and $p_{\alpha,r} \in A$ such that*

$$x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}, \tag{5}$$

where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. Moreover, if r is left invertible, then r_α is left invertible.

(b) *For every $x^\alpha, x^\beta \in Mon(A)$ there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that*

$$x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}, \tag{6}$$

where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.

In addition, the skew PBW extensions are filtered rings and its associated graded ring satisfies an interesting property, as shown in the following statement.

Proposition 2.4. *Let A be an arbitrary skew PBW extension of R. Then, A is a filtered ring with filtration given by*

$$F_m := \begin{cases} R & \text{if } \; m = 0 \\ \{f \in A \mid \deg(f) \leq m\} & \text{if } \; m \geq 1 \end{cases} \tag{7}$$

and the corresponding graded ring $Gr(A)$ is a quasi-commutative skew PBW extension of R. Moreover, if A is bijective, then $Gr(A)$ is a quasi-commutative bijective skew PBW extension of R.

Proof. See [7], Theorem 2.2. ☑

The above proposition enables us proving the Hilbert basis theorem for bijective skew PBW extensions.

Proposition 2.5 (Hilbert Basis Theorem)**.** *Let A be a bijective skew PBW extension of R. If R is a left (right) Noetherian ring then A is also a left (right) Noetherian ring.*

Proof. See [7], Corollary 2.4. ☑

***Remark* 2.6.** We developed the Gröbner bases theory for any bijective skew PBW extension. Specifically, we established a Buchberger's algorithm for these rings, the computation of syzygies module, as well as some applications as membership problem, calculation of intersections, quotients, presentation of a module, computing free resolutions, the kernel and image of an homomorphism (see Chapter 5 and Chapter 6 in [2], or [3]). In [6] where presented some other applications of this noncommutative Gröbner theory. In all of these works the theory and the applications have been illustrated with many examples.

3. For left ideals

In [7] was showed that if A is a skew PBW extension, then its associated graded ring $Gr(A)$ is a quasi-commutative skew PBW extension (see Theorem 2.2 there in). In this section we will prove this fact using a different technique. Furthermore, we establish the transfer of Gröbner bases between A and $Gr(A)$, when A is a bijective skew PBW extension.

By (2.4), given A a skew PBW extension of the ring R, the collection of subsets $\{F_p(A)\}_{p\in\mathbb{Z}}$ of A defined by

$$F_p(A) := \begin{cases} 0, & \text{if } p \leq -1, \\ R, & \text{if } p = 0, \\ \{f \in A \mid \deg(lm(f)) \leq p\}, & \text{if } p \geq 1. \end{cases}$$

is a filtration for the ring A, named *standard filtration.*

Now, notice that

$$F_p(A) = \{\sum c_\alpha x^\alpha \mid c_\alpha \in R \setminus \{0\},\, x^\alpha \in Mon(A),\, \deg(x^\alpha) \leq p\};$$

in this case, we say that this filtration is the *filtration $Mon(A)$-standard on A*. Moreover,

$$Mon(A) = \bigcup_{p\geq 0} Mon(A)_p,$$

where $Mon(A)_p := \{x^\alpha \in Mon(A) \mid \deg(x^\alpha) \leq p\}$, and if $|\alpha| = p$, then $x^\alpha \notin Mon(A)_{p-1}$. In this case, it says that $Mon(A)$ is a *strictly filtered basis.*

It can be noted that any filtration $\{F_p(A)\}_{p\in\mathbb{Z}}$ on A defines an order function $v : A \to \mathbb{Z}$ in the following way:

$$v(f) := \begin{cases} p, & \text{if } f \in F_p(A) - F_{p-1}(A), \\ -\infty, & \text{if } f \in \cap_{p\in\mathbb{Z}} F_p(A). \end{cases}$$

Definition 3.1. Let $Gr(A)$ be the graded ring associated to the filtered ring A, and let $f \in A$ with $f = \sum_{|\alpha|\leq p} c_\alpha x^\alpha$, where p =deg(f), $c_\alpha \in R \setminus \{0\}$

y $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. In what follows, $\eta(f)$ will denote *the image* (or *principal symbol*) of f in $Gr(A)$, i.e.,

$$\eta(f) := \sum_{|\alpha|=p} c_\alpha x^\alpha + F_{p-1}(A) \in F_p(A)/F_{p-1}(A).$$

Lemma 3.2. *Let A, $Mon(A)$ and $\{F_p(A)\}_p$ as above, then:*

(i) *For each $f \in A$, $deg(f) = v(f)$.*

(ii) *For each $p \in \mathbb{N}$, $Mon(A)_p$ is a R-basis for $F_p(A)$.*

(iii) *For x^α, $x^\beta \in Mon(A)$, $\eta(x^\alpha) = \eta(x^\beta)$ if and only if $x^\alpha = x^\beta$.*

Proof. (i) From definition of $\{F_p(A)\}_{p\in\mathbb{Z}}$ it follows that if $0 \neq f \in A$, then there exists $p \in \mathbb{N}$ such that $f \in F_p(A) - F_{p-1}(A)$ and, therefore, $v(f) = p$. But, if $f \in F_p(A) - F_{p-1}(A)$, then $\deg(f) = p$ and we obtain the equality.
(ii) Let $f \in F_p(A)$, then $f = \sum_{|\alpha|\leq p} c_\alpha x^\alpha$, and hence, $f \in {}_R\langle Mon(A)_p\}$. The linear independence of $Mon(A)_p$ it follows from fact that $Mon(A)_p \subseteq Mon(A)$ and $Mon(A)$ is linearly independent.
(iii) Let x^α, $x^\beta \in Mon(A)$ such that $0 \neq \eta(x^\alpha) = \eta(x^\beta) \in Gr(A)_p = F_p(A)/F_{p-1}(A)$; this last implies that $x^\alpha - x^\beta \in F_{p-1}(A)$, i.e., $x^\alpha - x^\beta \in {}_R\langle Mon(A)_{p-1}\}$. Now, since x^α, $x^\beta \notin F_{p-1}(A)$, we have that $x^\alpha - x^\beta = 0$, namely $x^\alpha = x^\beta$. The other implication is straightforward. ☑

Lemma 3.3. *If x^α, $x^\beta \in Mon(A)$, with $deg(x^\alpha) = p$ and $deg(x^\beta) = q$, then $\eta(x^\alpha x^\beta) = \eta(x^\alpha)\eta(x^\beta)$. In particular, if $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in F_p(A) - F_{p-1}(A)$, necessarily $\eta(x^\alpha) \neq 0$ and $\eta(x^\alpha) = \eta(x_1)^{\alpha_1} \cdots \eta(x_n)^{\alpha_n} \in Gr(A)_p$.*

Proof. In fact, $x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}$, where $c_{\alpha,\beta} \in R$ is left invertible and $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta| = p + q$ (see Theorem 2.3), whence $0 \neq \eta(x^\alpha x^\beta) = \overline{c_{\alpha,\beta} x^{\alpha+\beta}} = c_{\alpha,\beta}\overline{x^{\alpha+\beta}} \in F_{p+q}(A)/F_{p+q-1}(A)$. Furthermore, $0 \neq \eta(x^\alpha)\eta(x^\beta) = \overline{x^\alpha}\,\overline{x^\beta} = \overline{x^\alpha x^\beta} \in F_{p+q}(A)/F_{p+q-1}(A)$; but $x^\alpha x^\beta - c_{\alpha,\beta} x^{\alpha+\beta} = p_{\alpha,\beta} \in F_{p+q-1}(A)$, then $\overline{x^\alpha x^\beta} = \overline{c_{\alpha,\beta} x^{\alpha+\beta}}$, i.e., $\eta(x^\alpha x^\beta) = \eta(x^\alpha)\eta(x^\beta)$. ☑

Proposition 3.4. *Let A, $Mon(A)$ and $\{F_p(A)\}$ as before, then $\eta(Mon(A)_p) := \{\eta(x^\alpha) \mid x^\alpha \in Mon(A)_p\}$, forms a R-basis of $Gr(A)_p$ for each $p \in \mathbb{N}$. Moreover, $\eta(Mon(A)) := \{\eta(x^\alpha) \mid x^\alpha \in Mon(A)\}$ is a R-basis for $Gr(A)$.*

Proof. Let $f \in F_p(A) \setminus F_{p-1}(A)$, then $f = \sum_{|\alpha|\leq p} c_\alpha x^\alpha$ with $c_\alpha \in R \setminus \{0\}$ y $\eta(f) = \sum_{|\alpha|=p} c_\alpha \eta(x^\alpha) \neq 0$. By Lemma 3.3, $\eta(x^\alpha) \in Gr(A)_p$ for every α with $|\alpha| = p$, thus $\eta(Mon(A)_p)$ is a generating set for the left R-module $Gr(A)_p$. Now, suppose that there are $\lambda_i \in R$ such that $0 = \sum \lambda_i \eta(x^{\alpha_i}) \in Gr(A)_p$ for certain $x^{\alpha_i} \in Mon(A)_p$, then $\sum \lambda_i x^{\alpha_i} \in F_{p-1}(A)$; but $\deg(x^{\alpha_i}) = p$ for each i and $Mon(A)$ is a R-basis filtered strictly, hence $\lambda_i = 0$ for every i. ☑

The above preliminaries enables us to establish one of the main theorems of this section.

Theorem 3.5. *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a (bijective) skew PBW extension of ring R, then $Gr(A)$ is a (bijective) quasi-commutative skew PBW extension of R.*

***Proof*.** We must show that in $Gr(A)$ there exist nonzero elements $y_1, \dots, y_n$ satisfying the conditions in (a) from Definition 2.2. Define $y_i := \eta(x_i)$ for each $1 \leq i \leq n$; by Proposition 3.4 we have that

$$\eta(Mon(A)) := \{\eta(x^\alpha) = \eta(x_1)^{\alpha_1} \cdots \eta(x_n)^{\alpha_n} \mid x^\alpha \in Mon(A)\}$$

is a R-basis for $Gr(A)$. Now, given $r \in R \setminus \{0\}$, there is $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r}x_i = p_{i,r} \in R$; from last equality it follows that $\eta(x_i r) - \eta(c_{i,r}x_i) = \eta(p_{i,r}) = 0$, i.e., $\eta(x_i r) = \eta(c_{i,r}x_i) = c_{i,r}\eta(x_i)$; but $x_i r \neq 0$ for any nonzero $r \in R$ because $Mon(A)$ is a R-basis for the right R-module A_R (see [7], Proposition 1.7), thus $\eta(x_i r) = \eta(x_i)\eta(r) = \eta(x_i)r$, and consequently $\eta(x_i)r = c_{i,r}\eta(x_i)$. On the other hand, given $i, j \in \{1, \dots n\}$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - c_{i,j}x_i x_j = p_{i,j} \in R + Rx_1 + \cdots + Rx_n$; hence we have that $\eta(x_j x_i) = \eta(c_{i,j}x_i x_j) = c_{i,j}\eta(x_i)\eta(x_j)$, and by Lemma 3.3 $\eta(x_j x_i) = \eta(x_j)\eta(x_i)$, therefore $\eta(x_j)\eta(x_i) = c_{i,j}\eta(x_i)\eta(x_j)$. Since the $c_{i,r}$'s and $c_{i,j}$'s that define to $Gr(A)$ as a quasi-commutative skew PBW extension are the same that define A as a skew PBW extension of R, then the bijectivity of A implies the of $Gr(A)$. ☑

***Remark* 3.6.** The last theorem will allow us to establish a back and forth between Gröbner bases theory for A and $Gr(A)$. As we will show, the existence of one theory implies the existence of the other.

In the following, the set $\eta(Mon(A))$ will be denoted by $Mon(Gr(A))$. Thus, $Mon(Gr(A))$ is the basis for the left R-module $Gr(A)$ composed by the standard monomials in the variables $\eta(x_1), \dots, \eta(x_n)$.

We recall the definition of monomial order on a ring.

Definition 3.7. Let $\succeq$ be a total order on $Mon(A)$, it says that $\succeq$ is a monomial order on $Mon(A)$ if the following conditions hold:

(i) For every $x^\beta, x^\alpha, x^\gamma, x^\lambda \in Mon(A)$

$$x^\beta \succeq x^\alpha \Rightarrow lm(x^\gamma x^\beta x^\lambda) \succeq lm(x^\gamma x^\alpha x^\lambda).$$

(ii) $x^\alpha \succeq 1$, for every $x^\alpha \in Mon(A)$.

(iii) $\succeq$ is degree compatible, i.e., $|\beta| \geq |\alpha| \Rightarrow x^\beta \succeq x^\alpha$.

Monomial orders are also called *admissible orders*.

Proposition 3.8. *If $\succeq$ is a monomial order on $Mon(A)$, then relation $\succeq_{gr}$ defined over $Mon(Gr(A))$ by*

$$\eta(x^\alpha) \succeq_{gr} \eta(x^\beta) \Leftrightarrow x^\alpha \succeq x^\beta \tag{8}$$

is a monomial order for $Mon(Gr(A))$.

Proof. We will show that $\succeq_{gr}$ satisfies the conditions in the Definition 3.7: (i) Let $\eta(x^\alpha)$, $\eta(x^\beta)$, $\eta(x^\lambda)$, $\eta(x^\gamma) \in Mon(Gr(A))$ and suppose that $\eta(x^\beta) \succeq_{gr} \eta(x^\alpha)$, then,

$$lm(\eta(x^\gamma)\eta(x^\beta)\eta(x^\lambda)) \succeq_{gr} lm(\eta(x^\gamma)\eta(x^\alpha)\eta(x^\lambda)) \Leftrightarrow lm(\eta(x^\gamma x^\beta x^\lambda)) \succeq_{gr} lm(\eta(x^\gamma x^\alpha x^\lambda)).$$

But, $\eta(lm(x^\gamma x^\beta x^\lambda)) = lm(\eta(x^\gamma x^\beta x^\lambda))$ for all $\gamma, \beta, \lambda \in \mathbb{N}^n$: indeed, $\eta(x^\gamma x^\beta x^\lambda) = \overline{cx^{\gamma+\beta+\lambda}}$
$= c\eta(x^{\gamma+\beta+\lambda})$, where $c := c_{\gamma,\beta}c_{\gamma+\beta,\lambda}$ (see Remark 8 in [4]). Therefore,

$$lm(\eta(x^\gamma x^\beta x^\lambda)) = lm(c\eta(x^{\gamma+\beta+\lambda})) = \eta(x^{\gamma+\beta+\lambda}) = \eta(lm(x^\gamma x^\beta x^\lambda)).$$

Since $\succeq$ is a order monomial on $Mon(A)$, it has $lm(x^\gamma x^\beta x^\lambda) \succeq lm(x^\gamma x^\alpha x^\lambda)$, so that $\eta(lm(x^\gamma x^\beta x^\lambda)) \succeq_{gr} \eta(lm(x^\gamma x^\alpha x^\lambda))$, i.e., $lm(\eta(x^\gamma x^\beta x^\lambda)) \succeq_{gr} lm(\eta(x^\gamma x^\alpha x^\lambda))$. In consequence, $lm(\eta(x^\gamma)\eta(x^\beta)\eta(x^\lambda)) \succeq_{gr} lm(\eta(x^\gamma)\eta(x^\alpha)\eta(x^\lambda))$.

The conditions (ii) y (iii) in Definition 3.7 are easily verifiable. ☑

Lemma 3.9. *Let A as before, $\succeq$ a monomial order on $Mon(A)$ and $f \in A$ an arbitrary element. Then,*

(i) *$f \in F_p(A)$ if and only if $deg(f) \leq p$. Further, $f \in F_p(A) - F_{p-1}(A)$ if, and only, if $deg(f) = p$.*

(ii) *$\eta(lm(f)) = lm(\eta(f))$.*

Proof. (i) It follows from the definition of $F_p(A)$ and Lemma 3.2.

(ii) Let f be a nonzero polynomial in A; there exists $p \in \mathbb{N}$ such that $f \in F_p(A) - F_{p-1}(A)$. Let $f = \sum_{i=1}^n \lambda_i x^{\alpha_i}$, with $\lambda_i \in R\backslash\{0\}$ y $x^{\alpha_i} \in Mon(A)_p$, $1 \leq i \leq n$, where $x^{\alpha_1} \succ x^{\alpha_2} \succ \cdots \succ x^{\alpha_n}$. Hence, $lm(f) = x^{\alpha_1}$, $\deg(f) = p$ and $\eta(f) = \sum_{|\alpha_i|=p} \lambda_i \eta(x^{\alpha_i})$. From the definition given for $\succeq_{gr}$, we have that $lm(\eta(f)) = \eta(x^{\alpha_1}) = \eta(lm(f))$. ☑

We will prove that the reciprocal of the Proposition 3.8 also holds.

Proposition 3.10. *Let A and $Gr(A)$ as before. If $\succeq_{gr}$ is a monomial order on $Mon(Gr(A))$, then the relation $\succeq$ defined as*

$$x^\alpha \succ x^\beta \Leftrightarrow \eta(x^\alpha) \succ_{gr} \eta(x^\beta) \tag{9}$$

is a monomial order over $Mon(A)$.

Proof. Since $\succeq_{gr}$ is a well order, from Definition 9 it follows that $\succeq$ is a well order too. Now, we show that $\succeq$ is a monomial order: indeed, let x^α, x^β, x^γ, $x^\lambda \in Mon(A)$ and suppose that $x^\beta \succeq x^\alpha$, so:

$$\begin{cases} \eta(x^\beta) \succeq \eta(x^\alpha) \\ \eta(lm(x^\gamma x^\beta x^\lambda)) = lm(\eta(x^\gamma x^\beta x^\lambda)) = lm(\eta(x^\gamma)\eta(x^\beta)\eta(x^\lambda)) \\ \eta(lm(x^\gamma x^\alpha x^\lambda)) = lm(\eta(x^\gamma x^\alpha x^\lambda)) = lm(\eta(x^\gamma)\eta(x^\alpha)\eta(x^\lambda)) \\ lm(\eta(x^\gamma)\eta(x^\beta)\eta(x^\lambda)) \succeq_{gr} lm(\eta(x^\gamma)\eta(x^\alpha)\eta(x^\lambda)), \end{cases}$$

and hence, $lm(x^\gamma x^\beta x^\lambda) \succeq lm(x^\gamma x^\alpha x^\lambda)$. Clearly $x^\alpha \succeq 1$ for all $x^\alpha \in Mon(A)$, and $\succeq$ is degree compatible. ☑

Definition 3.11. Let I be a left (right or two side) ideal of A. The *graduation* of I (or the *associated graded ideal* to I) is defined as $G(I) := \oplus_p Gr(I)_{p\in\mathbb{N}}$, where $Gr(I)_p := I \cap F_p(A)/I \cap F_{p-1}(A) \cong (I + F_{p-1}(A)) \cap F_p(A)/F_{p-1}(A)$, for each $p \in \mathbb{N}$; (e.g., see [9]).

Before proceeding, let us recall the definition of Gröbner basis.

Definition 3.12. Let $I \neq 0$ be a left ideal of A and let G be a non empty finite subset of non-zero polynomials of I, we say that G is a Gröbner basis for I if each element $0 \neq f \in I$ is reducible w.r.t. G.

We have the following characterization for Gröbner bases.

Theorem 3.13. *Let $I \neq 0$ be a left ideal of A and let G be a finite subset of non-zero polynomials of I. Then the following conditions are equivalent:*

(i) *G is a Gröbner basis for I.*

(ii) *For any polynomial $f \in A$,*

$$f \in I \text{ if and only if } f \xrightarrow{G}_{+} 0.$$

(iii) *For any $0 \neq f \in I$ there exist $g_1, \dots, g_t \in G$ such that $lm(g_j)|lm(f)$, $1 \leq j \leq t$, (i.e., there exist $\alpha_j \in \mathbb{N}^n$ such that $\alpha_j + \exp(lm(g_j)) = \exp(lm(f))$) and*

$$lc(f) \in \langle \sigma^{\alpha_1}(lc(g_1))c_{\alpha_1,g_1}, \dots, \sigma^{\alpha_t}(lc(g_t))c_{\alpha_t,g_t}\}.$$

Proof. See [4], Theorem 24. ☑

Theorem 3.14. *Let A, $Gr(A)$, $Mon(A)$ and $Mon(Gr(A))$ as before, $\succeq$ a monomial order over $Mon(A)$, and I a left ideal of A. If $\overline{\mathcal{G}} = \{G_j\}_{j\in J}$ is a Gröbner basis for $Gr(I)$, with respect to the monomial order $\succeq_{gr}$, and such basis is formed by homogeneous elements, then $\mathcal{G} := \{g_j\}_{j\in J}$ is a Gröbner basis for I, where $g_j \in I$ is a selected polynomial with property that $\eta(g_j) = G_j$ for each $j \in J$.*

Proof. Let $0 \neq f \in I \cap F_p(A) \setminus F_{p-1}(A)$; we shall show that the condition (iii) in the Theorem 3.13 is satisfied: let $\overline{f} := \eta(f)$, then $0 \neq \overline{f} \in G(I)_p$. Since $\overline{\mathcal{G}}$ is a Gröbner basis of $G(I)$, there exist $G_1, \dots, G_t \in \overline{\mathcal{G}}$ such that $lm(G_j) \mid lm(\overline{f})$ for each $1 \leq j \leq t$ and $lc(\overline{f}) \in \langle \sigma^{\alpha_1}(lc(G_1))c_{\alpha_1,G_1}, \dots, \sigma^{\alpha_t}(lc(G_t))c_{\alpha_t,G_t}\}$, with $\alpha_j \in \mathbb{N}^n$ such that $\alpha_j + exp(lm(G_j)) = exp(lm(\overline{f})) = exp(lm(f)) = p$ and c_{α_j,G_j} is the coefficient determined by the product $\eta(x)^{\alpha_j} lm(G_j)$ in $Gr(A)$, for $1 \leq j \leq t$. From this last it follows that $lm(\eta(x)^{\alpha_j} lm(G_j)) = lm(\overline{f})$; but $lm(\eta(x)^{\alpha_j} lm(G_j)) = lm(\eta(x^{\alpha_j} x^{\beta_j}))$, where $x^{\beta_j} := lm(g_j)$ y $g_j \in I \cap F_p(A)$ is such that $\eta(g_j) = G_j$. From Lemma 3.9 we get that $lm(\eta(x^{\alpha_j} x^{\beta_j})) = \eta(lm(x^{\alpha_j} x^{\beta_j})) \in F(A)_p/F(A)_{p-1}$, so that $\eta(lm(x^{\alpha_j} x^{\beta_j})) = lm(\overline{f}) = \eta(lm(f))$. The latter implies that $lm(x^{\alpha_j} x^{\beta_j}) - lm(f) \in F_{p-1}(A)$ and, therefore, $lm(x^{\alpha_j} x^{\beta_j}) = lm(f)$, i.e., $lm(g_j) \mid lm(f)$ for each $1 \leq j \leq t$. Further, $lc(h) = lc(\eta(h))$ for all $h \in A$, then $lc(f) \in \langle \sigma^{\alpha_1}(lc(g_1))c_{\alpha_1,g_1}, \dots, \sigma^{\alpha_t}(lc(g_t))c_{\alpha_t,g_t}\}$. ☑

In this way, a Gröbner basis of $Gr(I)$ can be transfer to a Gröbner basis of I. In particular, from a Gröbner basis of $Gr(I)$ we can get a set of generators for I. Reciprocally, whether we need obtain a generating set of $Gr(I)$ from one of $I = \langle f_1, \dots, f_r\}$, we could think that $Gr(I) = \langle \eta(f_1), \dots, \eta(f_r)\}$. Nevertheless, this affirmation in general is not true: in fact, let $A = A_2(\Bbbk)$, the second Weyl algebra, i.e., $A = \Bbbk[x_1, x_2][y_1, \frac{\partial}{\partial x_1}][y_2, \frac{\partial}{\partial x_2}]$ with its associated standard filtration, and consider the left ideal I generated by $f_1 = x_1 y_1$ and $f_2 = x_2 y_1^2 - y_1$. Note that $x_1 \in I$, since $x_1 = (t_2 x_1^2 - x_1) f_1 - (t_1 x_1 + 2) f_2$, but $\eta(x_1) \notin \langle \eta(f_1), \eta(f_2)\}$, where $\eta(f_1) = \eta(t_1)\eta(x_1) \in Gr(I)_1$ and $\eta(f_2) = \eta(t_2)\eta(x_1)^2 \in Gr(I)_2$ (see [8]). However, if $G = \{f_1, \dots, f_r\}$ is a Gröbner basis for I, we will show that $\eta(G) = \{\eta(f_1), \dots, \eta(f_r)\}$ is a Gröbner basis for $Gr(I)$ and, from this we will take a generating set for $Gr(I)$.

Theorem 3.15. *With notation as above, let $\mathcal{G} = \{g_i\}_{i \in J}$ be a Gröbner basis for a left ideal I of A. Then $\overline{\mathcal{G}} = \{\eta(g_i)\}_{i \in J}$ is a Gröbner basis of $Gr(I)$ consisting of homogeneous elements.*

Proof. Since $Gr(I)$ is a homogeneous ideal, it suffices to show that every nonzero homogeneous element $F \in Gr(I)$ satisfies the condition (iii) in the Theorem 3.13. Let $0 \neq F \in Gr(I)_p$, then $F = \eta(f)$ for some $f \in I \cap F_p(A) - I \cap F_{p-1}(A)$ and there exist $g_1, \dots, g_t \in \mathcal{G}$ with the property that $lm(g_i) \mid lm(f)$ and $lc(f) \in \langle \sigma^{\alpha_1}(lc(g_1))c_{\alpha_1,g_1}, \dots, \sigma^{\alpha_t}(lc(g_t))c_{\alpha_t,g_t}\}$, where $\alpha_i \in \mathbb{N}^n$ is such that $\alpha_i + \exp(g_i) = \exp(f)$ for each $1 \leq i \leq t$. By Lemma 3.9 we have that $\eta(lm(f)) = lm(\eta(f)) = lm(F)$, then $lm(\eta(g_i)) \mid lm(F)$. Further, since $lc(f) = lc(\eta(f)) = lc(F)$, it follows that $lc(F) \in \langle \sigma^{\alpha_1}(lc(\eta(g_1)))c_{\alpha_1,\eta(g_1)}, \dots, \sigma^{\alpha_t}(lc(\eta(g_t)))c_{\alpha_t,\eta(g_t)}\}$ and, in consequence $\overline{\mathcal{G}}$ is a Gröbner basis for $Gr(I)$. ☑

4. For modules

Similar results about the transfer of Gröbner bases between A and $Gr(A)$ can be proved in the case of modules. For this, let M be a submodule of the free module A^m, $m \geq 1$, where A is a bijective skew PBW extension of a ring R. Define the following collection of subsets of M:

$$F_p(M) := \{\boldsymbol{f} \in M \mid \deg(\boldsymbol{f}) \leq p\}. \tag{10}$$

It is not difficult to show that the collection $\{F_p(M)\}_{p\geq 0}$ given in 10 is a filtration for M, called the *natural filtration* on M. With this filtration we can define the graded module associated to M, which will be denoted by $Gr(M)$, in the following way: $Gr(M) := \oplus_{p\geq 0} F_p(M)/F_{p-1}(M)$; if $\boldsymbol{f} \in F_p(M) - F_{p-1}(M)$, then $\boldsymbol{f}$ is said to have degree p. Thus, we may associate to $\boldsymbol{f}$ its *principal symbol* $\eta(\boldsymbol{f}) := \boldsymbol{f} + F_{p-1}(M) \in G_p(M) = F_p(M)/F_{p-1}(M)$. The $Gr(A)$-structure is given by, via distributive laws, the following multiplication:

$$\eta(r)\eta(\boldsymbol{f}) := \begin{cases} \eta(r\boldsymbol{f}), & \text{if } r\boldsymbol{f} \notin F_{i+j-1}(M), \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

where $r \in F_i(A) - F_{i-1}(A)$ and $\boldsymbol{f} \in F_j(M) - F_{j-1}(M)$.

Notice that any filtration $\{F_p(M)\}_{p\in\mathbb{Z}}$ on M defines an order function $v : M \to \mathbb{Z}$ in the following way:

$$v(\boldsymbol{f}) := \begin{cases} p, & \text{if } \boldsymbol{f} \in F_p(M) - F_{p-1}(M), \\ -\infty, & \text{if } \boldsymbol{f} \in \cap_{p\in\mathbb{Z}} F_p(M). \end{cases}$$

Lemma 4.1. *Let A, M and $\{F_p(M)\}_p$ be as above. Then for each $\boldsymbol{f} \in M$, $deg(\boldsymbol{f}) = v(\boldsymbol{f})$.*

Proof. From definition of $\{F_p(M)\}_{p\geq 0}$, it follows that if $\boldsymbol{0} \neq \boldsymbol{f} \in M$, then there exists $p \in \mathbb{N}$ such that $\boldsymbol{f} \in F_p(M) - F_{p-1}(M)$ and, therefore, $v(\boldsymbol{f}) = p$. But, if $f \in F_p(M) - F_{p-1}(M)$, then $\deg(\boldsymbol{f}) = p$ and we obtain the equality. ☑

We have a version of the Proposition 3.8 for module case. For this, remember that the monomials in $Gr(A)^m$ are given by $\overline{\boldsymbol{X}} = \eta(\boldsymbol{X}) := \eta(x^\alpha)\overline{\boldsymbol{e}}_i$, where $\overline{\boldsymbol{e}}_i$ is a canonical vector of $Gr(A)^m$. We also recall the definition of monomial orders on $Mon(A^m)$.

Definition 4.2. A monomial order on $Mon(A^m)$ is a total order $\succeq$ satisfying the following three conditions:

(i) $lm(x^\beta x^\alpha)\mathbf{e}_i \succeq x^\alpha \mathbf{e}_i$, for every monomial $\mathbf{X} = x^\alpha \mathbf{e}_i \in Mon(A^m)$ and any monomial x^β in $Mon(A)$.

(ii) If $\mathbf{Y} = x^{\beta}\mathbf{e}_j \succeq \mathbf{X} = x^{\alpha}\mathbf{e}_i$, then $lm(x^{\gamma}x^{\beta})\mathbf{e}_j \succeq lm(x^{\gamma}x^{\alpha})\mathbf{e}_i$ for every monomial $x^{\gamma} \in Mon(A)$.

(iii) $\succeq$ is degree compatible, i.e., $\deg(\mathbf{X}) \geq \deg(\mathbf{Y}) \Rightarrow \mathbf{X} \succeq \mathbf{Y}$.

Proposition 4.3. *If $>$ is a monomial order on $Mon(A^m)$, then relation $>_{gr}$ defined over $Mon(Gr(A)^m)$ by*

$$\eta(\boldsymbol{X}) >_{gr} \eta(\boldsymbol{Y}) \Leftrightarrow \boldsymbol{X} > \boldsymbol{Y} \tag{12}$$

is a monomial order for $Mon(Gr(A)^m)$.

Proof. We will show that $\succeq_{gr}$ satisfies the conditions in the Definition 4.2: to begin, note that $>_{gr}$ is a total order because $>$ it is. Now, to prove (i) we must show that $lm(\eta(x^{\beta})\eta(x^{\alpha}))\overline{\boldsymbol{e}}_i \geq_{gr} \eta(x^{\alpha})\overline{\boldsymbol{e}}_i$ for every $\overline{\boldsymbol{X}} = \eta(x^{\alpha})\overline{\boldsymbol{e}}_i \in Mon(Gr(A)^m)$ and $\eta(x^{\beta}) \in Mon(Gr(A))$. It can be noted that,

$$lm(\eta(x^{\beta})\eta(x^{\alpha}))\overline{\boldsymbol{e}}_i \geq_{gr} \eta(x^{\alpha})\overline{\boldsymbol{e}}_i \Leftrightarrow \eta(lm(x^{\beta}x^{\alpha}))\overline{\boldsymbol{e}}_i \geq_{gr} \eta(x^{\alpha})\overline{\boldsymbol{e}}_i.$$

Since $>$ is a monomial order on $Mon(A^m)$, we have that $lm(x^{\beta}x^{\alpha})\boldsymbol{e}_i \geq x^{\alpha}\boldsymbol{e}_i$ and, from (12) it follows that $\eta(lm(x^{\beta}x^{\alpha}))\overline{\boldsymbol{e}}_i \geq_{gr} \eta(x^{\alpha})\overline{\boldsymbol{e}}_i$. So, $lm(\eta(x^{\beta})\eta(x^{\alpha}))\overline{\boldsymbol{e}}_i \geq_{gr} \eta(x^{\alpha})\overline{\boldsymbol{e}}_i$.

For (ii), let $\overline{\boldsymbol{Y}} = \eta(x^{\beta})\overline{\boldsymbol{e}}_j$ and $\overline{\boldsymbol{X}} = \eta(x^{\alpha})\overline{\boldsymbol{e}}_i$ monomials in $Mon(Gr(A)^m)$ such that $\overline{\boldsymbol{Y}} \geq_{gr} \overline{\boldsymbol{X}}$. Given $\eta(x^{\gamma}) \in Mon(Gr(A))$, we have

$$\begin{aligned} lm(\eta(x^{\gamma})\eta(x^{\beta}))\overline{\boldsymbol{e}}_j \geq_{gr} lm(\eta(x^{\gamma})\eta(x^{\alpha}))\overline{\boldsymbol{e}}_i \Leftrightarrow \eta(lm(x^{\gamma}x^{\beta}))\overline{\boldsymbol{e}}_j \geq_{gr} \\ \eta(lm(x^{\gamma}x^{\alpha}))\overline{\boldsymbol{e}}_i. \end{aligned}$$

In $Mon(A)$ we get that $lm(x^{\gamma}x^{\beta})\boldsymbol{e}_j \geq lm(x^{\gamma}x^{\alpha})\boldsymbol{e}_i$ and, once again, from (12) it follows that $\eta(lm(x^{\gamma}x^{\beta}))\overline{\boldsymbol{e}}_j \geq_{gr} \eta(lm(x^{\gamma}x^{\alpha}))\overline{\boldsymbol{e}}_i$.

Finally is easily verifiable that $\geq_{gr}$ is degree compatible. ☑

Lemma 4.4. *Let A, M, $Gr(A)$, $Gr(M)$ and $<$ be as before, and consider an arbitrary element $\boldsymbol{f} \in M$. Then,*

(i) *$\boldsymbol{f} \in F_p(M)$ if, and only if, $deg(\boldsymbol{f}) \leq p$. Further, $\boldsymbol{f} \in F_p(M) - F_{p-1}(M)$ if, and only, if $deg(\boldsymbol{f}) = p$.*

(ii) *$\eta(lm(\boldsymbol{f})) = lm(\eta(\boldsymbol{f}))$.*

Proof. (i) It follows from the definition of $F_p(M)$ and Lemma 4.1. (ii) Let $\boldsymbol{f}$ be a nonzero vector in M, then there exists $p \in \mathbb{N}$ such that $\boldsymbol{f} \in F_p(M) - F_{p-1}(M)$. Thus, $\boldsymbol{f} = \sum_{i=1}^{l} \lambda_i \boldsymbol{X}_i$ with $\lambda_i \in R \setminus \{0\}$, $\boldsymbol{X}_i \in Mon(A^m)$ where $\deg(\boldsymbol{X}_i) \leq p$ for each $1 \leq i \leq l$, and $\boldsymbol{X}_1 > \boldsymbol{X}_2 > \cdots > \boldsymbol{X}_l$. Whence, $lm(\boldsymbol{f}) = \boldsymbol{X}_1$ and since $\deg(\boldsymbol{f}) = p$ and $\eta(\boldsymbol{f}) = \sum_{|\exp(\boldsymbol{X}_i)|=p} \lambda_i \eta(\boldsymbol{X}_i)$, from the definition given for $\geq_{gr}$, we have that $lm(\eta(\boldsymbol{f})) = \eta(\boldsymbol{X}_1) = \eta(lm(\boldsymbol{f}))$. ☑

The conversely of Proposition 4.3 is also true, as it is shown below.

Proposition 4.5. *With the same notation used so far, if $\geq_{gr}$ a monomial order on $Mon(Gr(A)^m)$, then $\geq$ defined as*

$$\boldsymbol{X} \geq \boldsymbol{Y} \Leftrightarrow \eta(\boldsymbol{X}) \geq_{gr} \eta(\boldsymbol{Y}) \tag{13}$$

is a monomial order over $Mon(A^m)$.

Proof. Since $\geq_{gr}$ is a total order, from Definition 13 it follows that $\geq$ is a total order also. Now, we show that $\geq$ is a monomial order: indeed, let $x^\beta \in Mon(A)$ and $\boldsymbol{X} = x^\alpha \boldsymbol{e}_i$ an element in $Mon(A^m)$; we must to show $lm(x^\gamma x^\beta)\boldsymbol{e}_i \geq x^\alpha \boldsymbol{e}_i$ for all $x^\gamma \in Mon(A)$; however

$$\eta(lm(x^\gamma x^\beta))\overline{\boldsymbol{e}}_i \geq \eta(x^\alpha)\overline{\boldsymbol{e}}_i \Leftrightarrow lm(\eta(x^\gamma)\eta(x^\beta))\overline{\boldsymbol{e}}_i \geq x^\alpha \overline{\boldsymbol{e}}_i$$

and since $\geq_{gr}$ is a monomial order, this last inequality is true. From (13) it follows that $lm(x^\gamma x^\beta)\boldsymbol{e}_i \geq x^\alpha \boldsymbol{e}_i$, as we had to show. Now, if $\boldsymbol{Y} = x^\beta \boldsymbol{e}_j$ and $\boldsymbol{X} = x^\alpha \boldsymbol{e}_i$ are monomials in $Mon(A^m)$ such that $\boldsymbol{Y} \geq \boldsymbol{X}$, then $\eta(\boldsymbol{Y}) \geq_{gr} \eta(\boldsymbol{X})$. Thus, given $\eta(x^\gamma) \in Mon(Gr(A))$ we have that

$$lm(\eta(x^\gamma)\eta(x^\beta))\overline{\boldsymbol{e}}_j \geq_{gr} lm(\eta(x^\gamma)\eta(x^\alpha))\overline{\boldsymbol{e}}_i$$

i.e.,

$$\eta(lm(x^\gamma x^\beta))\overline{\boldsymbol{e}}_j \geq_{gr} \eta(lm(x^\gamma x^\alpha))\overline{\boldsymbol{e}}_i.$$

This implies that $lm(x^\gamma x^\beta)\boldsymbol{e}_j \geq lm(x^\gamma x^\alpha)\boldsymbol{e}_i$. Finally, it is easy to prove that $\geq$ is degree compatible. ☑

We are ready to prove the main theorem of this last section.

Theorem 4.6. *Let A, $Gr(A)$, $Mon(A)$ and $Mon(Gr(A))$ be as before, $\geq$ a monomial order over $Mon(A^m)$, and M a nonzero submodule of A^m. The following statements hold:*

(i) *If $\overline{\mathcal{G}} = \{\boldsymbol{G}_j\}_{j\in J}$ is a Gröbner basis for $Gr(M)$, with respect to the monomial order $\geq_{gr}$, and such basis is formed by homogeneous elements, then $\mathcal{G} := \{\boldsymbol{g}_j\}_{j\in J}$ is a Gröbner basis for M, where $\boldsymbol{g}_j \in M$ is a selected vector with the property that $\eta(\boldsymbol{g}_j) = \boldsymbol{G}_j$ for each $j \in J$.*

(ii) *If $\mathcal{G} = \{\boldsymbol{g}_i\}_{i\in J}$ is a Gröbner basis for M, then $\overline{\mathcal{G}} = \{\eta(\boldsymbol{g}_i)\}_{i\in J}$ is a Gröbner basis of $Gr(M)$ consisting of homogeneous elements.*

Proof. (i) Let $\mathbf{0} \neq \boldsymbol{f} \in F_p(M) \setminus F_{p-1}(M)$; we shall show that the condition (iii) in Theorem 3.13, for module case, is satisfied (see [5], Theorem 26): let $\overline{\boldsymbol{f}} := \eta(\boldsymbol{f})$, then $\mathbf{0} \neq \overline{\boldsymbol{f}} \in G(M)_p$. Since $\overline{\mathcal{G}}$ is a Gröbner basis of $G(M)$, there exist $\boldsymbol{G}_1, \ldots, \boldsymbol{G}_t \in \overline{\mathcal{G}}$ such that $lm(\boldsymbol{G}_j) \mid lm(\overline{\boldsymbol{f}})$ for each $1 \leq j \leq t$ and $lc(\overline{\boldsymbol{f}}) \in \langle \sigma^{\alpha_1}(lc(\boldsymbol{G}_1))c_{\alpha_1,\boldsymbol{G}_1}, \ldots, \sigma^{\alpha_t}(lc(\boldsymbol{G}_t))c_{\alpha_t,\boldsymbol{G}_t}\}$, with $\alpha_j \in \mathbb{N}^n$ such that $\alpha_j + \exp(lm(\boldsymbol{G}_j)) = \exp(lm(\overline{\boldsymbol{f}})) = p$ and $c_{\alpha_j,\boldsymbol{G}_j}$ is the coefficient determined by the product $\eta(x)^{\alpha_j} lm(\boldsymbol{G}_j)$ in $Gr(M)$, for $1 \leq j \leq t$. But, $\exp(lm(\overline{\boldsymbol{f}})) = \exp(lm(\boldsymbol{f}))$, thus of the above mentioned follows that $lm(\eta(x^{\alpha_j})lm(\boldsymbol{G}_j)) = lm(\overline{\boldsymbol{f}})$; note that $lm(\eta(x^{\alpha_j})lm(\boldsymbol{G}_j)) = lm(\eta(x^{\alpha_j}\boldsymbol{X}_j))$, where $\boldsymbol{X} := lm(\boldsymbol{g}_j)$ and $\boldsymbol{g}_j \in F_p(M)$ is such that $\eta(\boldsymbol{g}_j) = \boldsymbol{G}_j$. From Lemma 4.4 we get that $lm(\eta(x^{\alpha_j}\boldsymbol{X})) = \eta(lm(x^{\alpha_j}\boldsymbol{X})) \in F(M)_p/F(M)_{p-1}$, so that $\eta(lm(x^{\alpha_j}\boldsymbol{X})) = lm(\overline{\boldsymbol{f}}) = \eta(lm(\boldsymbol{f}))$. The latter implies that $lm(x^{\alpha_j}\boldsymbol{X}) - lm(\boldsymbol{f}) \in F_{p-1}(M)$ and, therefore, $lm(x^{\alpha_j}\boldsymbol{X} = lm(\boldsymbol{f})$, i.e., $lm(\boldsymbol{g}_j) \mid lm(\boldsymbol{f})$ for each $1 \leq j \leq t$. Further, $lc(h) = lc(\eta(\boldsymbol{h}))$ for all $\boldsymbol{h} \in A^m$, then $lc(\boldsymbol{f}) \in \langle \sigma^{\alpha_1}(lc(\boldsymbol{g}_1))c_{\alpha_1,\boldsymbol{g}_1}, \ldots, \sigma^{\alpha_t}(lc(\boldsymbol{g}_t))c_{\alpha_t,\boldsymbol{g}_t}\}$.

(ii) Since $Gr(M)$ is a graded module, it suffices to show that every nonzero homogeneous element $\boldsymbol{F} \in Gr(M)$ satisfies the condition (iii) in the Theorem 3.13 for module case. Suppose that $\boldsymbol{F} \in Gr(M)_p$; then, $\boldsymbol{F} = \eta(\boldsymbol{f})$ for some $\boldsymbol{f} \in F_p(M) - F_{p-1}(M)$ and there exist $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_t \in \mathcal{G}$ with the property that $lm(\boldsymbol{g}_i) \mid lm(\boldsymbol{f})$ and $lc(\boldsymbol{f}) \in \langle \sigma^{\alpha_1}(lc(\boldsymbol{g}_1))c_{\alpha_1,\boldsymbol{g}_1}, \ldots, \sigma^{\alpha_t}(lc(\boldsymbol{g}_t))c_{\alpha_t,\boldsymbol{g}_t}\}$, where $\alpha_i \in \mathbb{N}^n$ is such that $\alpha_i + \exp(\boldsymbol{f}_i) = \exp(\boldsymbol{f})$ for each $1 \leq i \leq t$. By Lemma 4.4 we have that $lm(\boldsymbol{f}) = lm(\eta(\boldsymbol{f})) = lm(\boldsymbol{F})$, then $lm(\eta(\boldsymbol{g}_i)) \mid lm(\boldsymbol{F})$ and, since $lc(\boldsymbol{f}) = lc(\eta(\boldsymbol{f})) = lc(\boldsymbol{F})$, it follows that $lc(\boldsymbol{F}) \in \langle \sigma^{\alpha_1}(lc(\eta(\boldsymbol{g}_1)))c_{\alpha_1,\eta(\boldsymbol{g}_1)}, \ldots,$ $\sigma^{\alpha_t}(lc(\eta(\boldsymbol{g}_t)))c_{\alpha_t,\eta(\boldsymbol{g}_t)}\}$ and, hence, $\overline{\mathcal{G}}$ is a Gröbner basis for $Gr(M)$. ☑

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