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The Padovan numbers of the form $6^a \pm 6^b \pm 6^c$

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Abstract. Let $(P_n)_{n\geqslant 0}$ be the Padovan sequence given by $P_0 = 0, P_1 = P_2 = 1$ and the recurrence formula $P_{n+3} = P_{n+1} + P_n$ for all $n \geqslant 0$. In this note, we completely solve the Diophantine equation

$$P_n = 6^a \pm 6^b \pm 6^c$$

in non-negative integers (n, a, b, c) with $a \geqslant b \geqslant c \geqslant 0$.

Keywords: Padovan sequence, Linear forms in logarithms, Reduction method.

MSC2020: 11D45,11D61, 11J86.

Los números de Padovan de la forma $6^a \pm 6^b \pm 6^c$

Resumen. Sea $(P_n)_{n\geqslant 0}$ la sucesión de Padovan dada mediante $P_0 = 0, P_1 = P_2 = 1$ y la fórmula de recurrencia $P_{n+3} = P_{n+1} + P_n$ se satisface para todo $n \geqslant 0$. En este artículo se resuelve completamente la ecuación Diofántica

$$P_n = 6^a \pm 6^b \pm 6^c$$

en enteros no negativos (n, a, b, c) con $a \geqslant b \geqslant c \geqslant 0$.

Palabras clave: Sucesión de Padovan, Formas lineales en logarítmos, Método de reducción.

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1. Introduction

Let $(F_n)_{n\geqslant 0}$ be the *Fibonacci sequence*. It is given by the initial conditions $F_0 = 0$, $F_1 = 1$ and the recurrence formula

$$F_{n+2} = F_{n+1} + F_n$$

holds for all $n \geqslant 0$. Let's consider the Diophantine equation

$$F_n = x^a \pm x^b + 1 \tag{1}$$

in positive integers (n, x, a, b) with $\max\{a, b\} \geqslant 2$. The case $x = p$ with p being a prime number is studied in [9] by Luca and Szalay. They show that such an equation has only finitely many solutions. Then, the same conclusion is obtained by Laishram and Luca in [7] where this time x is of the form $p^c q^d$ where p and q are prime numbers. In [6], the particular case $x = 2$ was completely solved.

There are many other instances of Diophantine equations of the same flavour as the above one. For example, in [8] the Diophantine equation $F_n = p^a \pm p^b$, where p is prime number, is studied. And, in [12] the squares of the form $2^a \pm 2^b \pm 2^c$ are found.

Now, let us consider the *Padovan sequence* $(P_n)_{n\geqslant 0}$, named after the architect R. Padovan. It is the ternary recurrence sequence given by $P_0 = 0$, $P_1 = P_2 = 1$ and the recurrence formula

$$P_{n+3} = P_{n+1} + P_n, \qquad \text{holds for all} \quad n \geqslant 0.$$

This is A000931 sequence in [11]. Its first few terms are

$$0,\ 1,\ 1,\ 1,\ 2,\ 2,\ 3,\ 4,\ 5,\ 7,\ 9,\ 12,\ 16,\ 21,\ 28,\ 37,\ 49,\ 65,\ 86,\ 114,\ 151,\ \ldots$$

Motivated by the above problems, in this note we study the Diophantine equation

$$P_n = 6^a \pm 6^b \pm 6^c \tag{2}$$

in the non-negative integers (n, a, b, c) with $a \geqslant b \geqslant c \geqslant 0$. To avoid numerical repeated solutions we will assume that $n \neq 1, 2, 4$. Note that for all non-negative integer a, $(3, a, a, 0)$ is clearly a solution to the case $P_n = 6^a - 6^b + 6^c$ with $a = b$. Let us call these *trivial solutions* of equation (2). Our result is the following:

Theorem 1.1. *All non trivial solutions of equation* (2) *in non-negative integers* (n, a, b, c) *with* $n \neq 1, 2, 4$ *and* $a \geqslant b \geqslant c \geqslant 0$ *are*

$$P_6 = 6^0 + 6^0 + 6^0, \quad P_7 = 6^1 - 6^0 - 6^0, \quad P_{27} = 6^4 - 6^3 + 6^0, \quad P_{34} = 6^5 - 6^2 - 6^0.$$

2. Linear forms in logarithms, reduction method

In proving Theorem 1.1 we use lower bounds for linear forms in logarithms, and we use the result due to Matveev explained in Theorem 2.1. Let α be an algebraic number of degree d, let $a > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\alpha = \alpha^{(1)}, \dots, \alpha^{(d)}$ denote its conjugates. The *logarithmic height* of α is defined as

$$h(\alpha) = \frac{1}{d}\left(\log a + \sum_{i=1}^{d} \log \max\left\{\left|\alpha^{(i)}\right|, 1\right\}\right).$$

The basic properties of the function h are the following. For α, β algebraic numbers and $m \in \mathbb{Z}$ we have

- $h(\alpha + \beta) \leqslant h(\alpha) + h(\beta) + \log(2)$,
- $h(\alpha\beta) \leqslant h(\alpha) + h(\beta)$,
- $h(\alpha^m) = |m| h(\alpha)$.

Now, let $\mathbb{K}$ be a real number field of degree $d_{\mathbb{K}}$, $\alpha_1, \dots, \alpha_\ell \in \mathbb{K}$ positive elements and $b_1, \dots, b_\ell \in \mathbb{Z} \setminus \{0\}$. Let $B \geqslant \max\{|b_1|, \dots, |b_\ell|\}$ and

$$\Lambda = \alpha_1^{b_1} \cdots \alpha_\ell^{b_\ell} - 1.$$

Let $A_1, \dots, A_\ell$ be real numbers with

$$A_i \geqslant \max\{d_{\mathbb{K}}\, h(\alpha_i),\ |\log \alpha_i|,\ 0.16\} \quad \text{for} \quad i = 1, 2, \dots, \ell.$$

The following result is due to Matveev in [10] (see also Theorem 9.4 in [2]).

Theorem 2.1. *(Matveev's Theorem) Assume that $\Lambda \neq 0$. Then*

$$\log|\Lambda| > -1.4 \cdot 30^{\ell+3} \cdot \ell^{4.5} \cdot d_{\mathbb{K}}^2 \cdot (1 + \log d_{\mathbb{K}}) \cdot (1 + \log B) A_1 \cdots A_\ell.$$

In this paper we always use $\ell := 3$. Further $\mathbb{K} := \mathbb{Q}(\gamma)$, where γ is given at the beginning of Section 3, has degree $d_{\mathbb{K}} = 3$. So, once and for all we fix the constant

$$C := 1.4 \cdot 30^{3+3} \cdot 3^{4.5} \cdot 3^2 \cdot (1 + \log 3) \approx 2.70443 \times 10^{12}.$$

Our second tool is a version of the reduction method of Baker-Davenport based on the lemma in [1]. We shall use the following one given by Bravo, Gómez and Luca in [3] (see also [4]). For a real number x, we write $\|x\|$ for the distance from x to the nearest integer.

Lemma 2.2. *Let M be a positive integer. Let $\tau,\ \mu,\ A > 0,\ B > 1$ be given real numbers. Assume that p/q is a convergent of τ such that $q > 6M$ and $\varepsilon := \|\mu q\| - M\|\tau q\| > 0$. If (n, m, w) is a positive solution to the inequality*

$$0 < |n\tau - m + \mu| < \frac{A}{B^w}$$

with $n \leqslant M$ then

$$w < \frac{\log(Aq/\varepsilon)}{\log(B)}.$$

Finally, the following result of Guzmán and Luca [5] will be very useful.

Lemma 2.3. *If $m \geqslant 1$, $T > (4m^2)^m$ and $T > x/(\log x)^m$. Then*

$$x < 2^m T (\log T)^m.$$

3. Proof of Theorem 1.1

Let us to start with some basic properties of the Padovan sequence. For a complex number z we write $\overline{z}$ for its complex conjugate. Let $\omega \neq 1$ be a cubic root of 1. Put

$$\gamma := \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}}, \quad \delta := \omega\sqrt[3]{\frac{9+\sqrt{69}}{18}} + \overline{\omega}\sqrt[3]{\frac{9-\sqrt{69}}{18}}.$$

It is clear that $\gamma, \delta, \overline{\delta}$ are the roots of the $\mathbb{Q}$-irreducible polynomial $X^3 - X - 1$. We also have the Binet formula

$$P_n = c_1\gamma^n + c_2\delta^n + c_3\overline{\delta}^n, \tag{3}$$

which holds for all $n \geqslant 0$, where

$$c_1 = \frac{\gamma(\gamma+1)}{2\gamma+3}, \qquad c_2 = \frac{\delta(\delta+1)}{2\delta+3}, \qquad c_3 = \overline{c_2}. \tag{4}$$

Formula (3) follows from the general theorem on linear recurrence sequences since the above polynomial is the characteristic polynomial of the Padovan sequence. We note that

$$\gamma = 1.32471\ldots, \ |\delta| = 0.86883\ldots, \ c_1 = 0.54511\ldots, \ |c_2| = 0.28241\ldots$$

Further, the inequalities

$$\gamma^{n-3} \leqslant P_n \leqslant \gamma^{n-1}, \tag{5}$$

hold for all $n \geqslant 1$. These, formula (3) and inequalities (5) can be proved by induction.

Observe that the study of the non-trivial solutions of equation (2) reduces to the study of equations of the following form:

$$\begin{aligned} P_n &= t\cdot 6^a \text{ where } t \in \{1,\ 3\} \text{ and } a \geqslant 0; && (6)\\ P_n &= t\cdot 6^a \pm t_1\cdot 6^b \text{ where } t, t_1 \in \{1,2\}, t \neq t_1 \text{ and } \quad a > b \geqslant 0; && (7)\\ P_n &= 6^a \pm 6^b \pm 6^c \quad a > b > c \geqslant 0. && (8) \end{aligned}$$

An elementary analysis shows that the right hand side of each these equations is always positive. So, we assume $n \geqslant 1$. As $n \neq 1, 2, 4$, we assume throughout the proof that $n \geqslant 3$ with $n \neq 4$. The most involved case is equation (8), so we start with it.

3.1. Case (8)

Recall that $n \geqslant 3$, $n \neq 4$ and $a > b > c \geqslant 0$. From inequalities (5) we obtain

$$\gamma^{n-3} \leqslant P_n = 6^a \pm 6^b \pm 6^c < 6^{a+1} \quad \text{and} \quad \gamma^{n-1} \geqslant P_n = 6^a \pm 6^b \pm 6^c > 6^{a-2}.$$

So,

$$(n-3)\frac{\log\gamma}{\log 6} < a+1 \quad \text{and} \quad (n-1)\frac{\log\gamma}{\log 6} > a-2. \tag{9}$$

In particular note that $a \leqslant n$ since $(\log\gamma/\log 6) < 1$. Now, if $n \leqslant 500$ from (9) we see that $a \leqslant 80$. Running a computer basic program in the range $0 \leqslant n \leqslant 500$, $0 \leqslant c < b < a \leqslant 80$

we find the last two solutions written in Theorem 1.1. We will prove that these are all of them in this case.

From now on, we assume $n > 500$. In this case, (9) gives $a \geqslant 76$. The first task is to obtain an absolute upper bound on n. To this end, from the Binet formula (3) we rewrite our equation as

$$c_1\gamma^n - 6^a = \pm 6^b \pm 6^c - c_2\delta^n - c_3\overline{\delta}^n.$$

Dividing through by 6^a, we obtain

$$\left|c_1\gamma^n 6^{-a} - 1\right| < \frac{1}{6^{a-b-1}}. \tag{10}$$

Let Λ be the expression inside in the left hand side of (10). Now, if $\Lambda = 0$ then $c_1\gamma^n = 6^a$ and, by taking norms we conclude that the norm of c_1 is an integer which is not. The norm of c_1 is $1/23$. Hence $\Lambda \neq 0$ and we apply Matveev's inequality to it by taking

$$\alpha_1 = c_1, \alpha_2 = \gamma, \alpha_3 = 6, \quad b_1 = 1, b_2 = n, b_3 = -a.$$

Thus $B = n$. The heights $h(\alpha_2)$, $h(\alpha_3)$ are $\log\gamma/3$ and $\log 6$, respectively. For α_1 we use the properties of the height to conclude

$$h(\alpha_1) \leqslant \log\gamma + 5\log 2.$$

Thus, we take $A_1 = 11.3, A_2 = 0.3$, and $A_3 = 5.4$. Then,

$$\log|\Lambda| > -C \cdot (1 + \log n) \cdot 11.3 \cdot 0.3 \cdot 5.4.$$

Comparing this with (10), we obtain

$$(a-b)\log 6 < 4.95073 \times 10^{13}(1 + \log n). \tag{11}$$

Again, from the Binet formula (3) we rewrite (2) and obtain

$$|c_1\gamma^n - (6^{a-b} \pm 1)6^b| < 6^{c+1}.$$

Dividing through by $6^a \pm 6^b$ we get

$$\left|\frac{c_1}{6^{a-b} \pm 1}\gamma^n 6^{-b} - 1\right| < \frac{6^{c+1}}{6^a \pm 6^b} < \frac{1}{6^{a-c-2}}, \tag{12}$$

where we use $6^a \pm 6^b > 6^{a-1}$. Let Λ_1 be the expression inside of the absolute value on the left side of (12). With an argument as given for Λ above, we see that $\Lambda_1 \neq 0$ and we apply Matveev's to it. To do this, we consider

$$\alpha_1 = \frac{c_1}{6^{a-b} \pm 1}, \alpha_2 = \gamma, \alpha_3 = 6, \quad b_1 = 1, b_2 = n, b_3 = -b.$$

Thus, $B = n$. The heights of α_2 and α_3 are already calculated. For α_1, we again use the properties of the height and (11) to conclude that

$$h(\alpha_1) < h(c_1) + h(6^{a-b} \pm 1) < 4.95074 \times 10^{13}(1 + \log n).$$

So we take $A_1 = 1.48522 \times 10^{14}(1+\log n)$ and A_2, A_3 as above. Hence, from Matveev's inequality we obtain

$$\log|\Lambda_1| > -C\cdot(1+\log n)\cdot(1.48522\times 10^{14}(1+\log n))\cdot 0.3\cdot 5.4,$$

which compared with (12) yields

$$(a-c)\log 6 < 6.50701\times 10^{26}(1+\log n)^2. \tag{13}$$

In particular, since $b<a$, we also have an upper bound on $(b-c)\log 6$.

Finally, from the Binet formula (3) we rewrite again (2) and obtain

$$|c_1\gamma^n - (6^{a-c}\pm 6^{b-c}\pm 1)6^c| < 1.$$

Dividing through by $6^a \pm 6^b \pm 6^c$ we get

$$\left|\frac{c_1}{6^{a-c}\pm 6^{b-c}\pm 1}\gamma^n 6^{-c} - 1\right| < \frac{1}{6^a\pm 6^b\pm 6^c} < \frac{1}{6^{a-1}} < \frac{36}{\gamma^{n-3}} < \frac{1}{\gamma^{n-16}}, \tag{14}$$

where we use $6^{a+1} > \gamma^{n-3}$ from (9). Let Λ_2 be the expression inside of the absolute value on the left side of (14). As above, $\Lambda_2 \neq 0$ and we apply Matveev's inequality to it. Now, we consider

$$\alpha_1 = \frac{c_1}{6^{a-c}\pm 6^{b-c}\pm 1}, \alpha_2=\gamma, \alpha_3 = 6, \quad b_1 = 1, b_2 = n, b_3 = -c.$$

Thus, $B = n$. The heights of α_2 and α_3 are already calculated. For α_1, from (13) we have

$$h(\alpha_1) < h(c_1) + h(6^{a-c}\pm 6^{b-c}\pm 1) < 1.3014\times 10^{27}(1+\log n)^2.$$

So we take $A_1 = 3.9042\times 10^{27}(1+\log n)^2$ and A_2, A_3 as above. Hence, from Matveev's inequality we obtain

$$\log|\Lambda_2| > -C\cdot(1+\log n)\cdot(3.9042\times 10^{27}(1+\log n)^2)\cdot 0.3\cdot 5.4,$$

which compared with (14) yields

$$n\log\gamma < 1.7105\times 10^{40}(1+\log n)^3.$$

Thus, $n < 4.86629\times 10^{41}(\log n)^3$ and from Lemma 2.3 we get the following absolute upper bound on n:

$$n < 3.44305\times 10^{48}. \tag{15}$$

Now, the second step is to reduce this upper bound on n. To do this, we consider

$$\Gamma = n\log\gamma - a\log 6 + \log c_1,$$

and go to (10). Assume that $a-b>1$. Note that $e^\Gamma - 1 = \Lambda \neq 0$. Thus, $\Gamma\neq 0$. If $\Gamma > 0$, we have that

$$0 < \Gamma < e^\Gamma - 1 = |\Lambda| < \frac{1}{6^{a-b-1}}.$$

If on the other hand, $\Gamma < 0$, we then have that $1 - e^{\Gamma} = |\Lambda| < 1/2$. Thus, $e^{\Gamma} < 2$. Hence,

$$0 < |\Gamma| < e^{|\Gamma|} - 1 = e^{|\Gamma|}|\Lambda| < \frac{2}{6^{a-b-1}}.$$

Thus, in both cases, we have

$$0 < |\Gamma| < \frac{2}{6^{a-b-1}}.$$

Dividing through by $\log 6$, we obtain

$$0 < |n\tau - a + \mu| < \frac{7}{6^{a-b}},$$

where

$$\tau := \frac{\log \gamma}{\log 6} \qquad \text{and} \qquad \mu := \frac{\log c_1}{\log 6}.$$

Now, we apply Lemma 2.2. To do this, we take $M = 3.44305 \times 10^{48}$, which from (15) is the upper bound on n. With the help of *Mathematica*, we found that the convergent

$$\frac{p_{107}}{q_{107}} = \frac{600832652910285560285991594277621556411089705259445 5}{3828411183997692351030135749270278066621548397729669 8}$$

of τ is such that $q_{107} > 6M$ and $\varepsilon = ||q_{107} \cdot \mu|| - M||q_{107} \cdot \tau|| = 0.284414 > 0$. Thus from Lemma 2.2, with $A := 7$ and $B := 6$, we obtain that

$$a - b < \frac{\log(7 \cdot q_{107}/\varepsilon)}{\log 6} < 70.$$

Now, consider

$$\Gamma_1 = n \log \gamma - b \log 6 + \log\left(\frac{c_1}{6^{a-b} \pm 1}\right),$$

and go to (12). Assume that $a - c > 2$. Note that $e^{\Gamma_1} - 1 = \Lambda_1 \neq 0$. Thus, $\Gamma_1 \neq 0$. As in the above case by considering again the cases $\Gamma_1 > 0$ and $\Gamma_1 < 0$ we conclude that

$$0 < |\Gamma_1| < \frac{2}{6^{a-c-2}}.$$

Dividing through by $\log 6$, we obtain

$$0 < |n\tau - b + \mu| < \frac{41}{6^{a-c}},$$

where τ is as above and

$$\mu := \frac{\log\left(c_1/(6^{a-b} \pm 1)\right)}{\log 6}.$$

Consider

$$\mu_k := \frac{\log\left(c_1/(6^k \pm 1)\right)}{\log 6}, \qquad k = 2, 3, \ldots, 69.$$

With *Mathematica* we find again that the $107-th$ convergent of τ is such that $q_{107} > 6M$ and $\varepsilon_k \geqslant 0.0162182$ for all $k = 2, \ldots, 69$. We calculated $\log(q_{107} \cdot 41/\varepsilon_k)/\log 6$ for all

$k = 2, \ldots, 69$ and found that the maximum of these values is at most 71. Therefore $a - c \leqslant 71$.

Finally, consider

$$\Gamma_2 = n \log \gamma - c \log 6 + \log \left(\frac{c_1}{6^{a-c} \pm 6^{b-c} \pm 1} \right),$$

and go to (14). Note that $e^{\Gamma_2} - 1 = \Lambda_2 \neq 0$. Thus, $\Gamma_2 \neq 0$. Again as above, we can conclude that

$$0 < |\Gamma_2| < \frac{2}{\gamma^{n-16}}.$$

Dividing through by $\log 6$, we obtain

$$0 < |n\tau - c + \mu| < \frac{101}{\gamma^n},$$

where, τ is as above and

$$\mu := \frac{\log(c_1/(6^{a-c} \pm 6^{b-c} \pm 1))}{\log 6}.$$

Consider

$$\mu_{j,k} := \frac{\log(c_1/(6^j \pm 6^k \pm 1))}{\log 6}, \qquad j = 3, \ldots, 71, \quad k < j.$$

Again, the 107-*th* convergent of τ is such that $q_{107} > 6M$ and $\varepsilon_{j,k} \geqslant 0.0000191955$ for all $j = 3, \ldots, 71$, $k < j$. Finally, by calculating $\log(q_{107} \cdot 101/\varepsilon_{j,k})/\log \gamma$ for all these cases we find that the maximum of these values is at most 485. Therefore $n \leqslant 485$ which contradicts the assumption on n and finish the proof of this case.

3.2. Case (7)

This case also follows the same lines of argument as in Case (8). So, we will not write all detailed calculations but only the result of the step.

As above, $n \geqslant 3$, $n \neq 4$; $t, t_1 \in \{1, 2\}, t \neq t_1$ and $a > b \geqslant 0$. The inequalities

$$\gamma^{n-3} \leqslant P_n = t \cdot 6^a \pm t_1 \cdot 6^b < 6^{a+1} \quad \text{and} \quad \gamma^{n-1} \geqslant P_n = t \cdot 6^a \pm t_1 \cdot 6^b > 6^{a-1},$$

where $t, t_1 \in \{1, 2\}$ and $t \neq t_1$, show that we can and we will use the same inequalities given in (9). In particular $a \leqslant n$ and with a basic computer program in the intervall $0 \leqslant n \leqslant 350$ and $0 \leqslant b < a \leqslant 55$ we obtain the solution $P_7 = 6^1 - 2 \cdot 6^0 = 6^1 - 6^0 - 6^0$ for this case written in Theorem 1.1. As above, we now show it is the only one.

Let $n > 350$. Thus $a > 53$. As above, from Binet's formula equation (7) gives

$$\left| \frac{c_1}{t} \gamma^n 6^{-a} - 1 \right| < \frac{1}{6^{a-b-1}}. \tag{16}$$

Let Λ be the expression inside of the absolute value on the left side of (16). Now, being $\Lambda \neq 0$ we take

$$\alpha_1 = \frac{c_1}{t}, \alpha_2 = \gamma, \alpha_3 = 6, \quad b_1 = 1, b_2 = n, b_3 = -a,$$

and apply Matveev's to it. So, $B = n$. We already know the heights of α_2 and α_3 and for α_1 we have

$$h(\alpha_1) < h(c_1) + h(t) < \log\gamma + 6\log 2.$$

So we take $A_1 = 13.4$ and $A_2 = 0.3, A_3 = 5.4$. Hence, from Matveev's inequality we get

$$\log|\Lambda_1| > -C \cdot (1 + \log n) \cdot 13.4 \cdot 0.3 \cdot 5.4,$$

which compared with (16) yields

$$(a - b)\log 6 < 5.87079 \times 10^{13}(1 + \log n). \tag{17}$$

From the Binet formula (3) we rewrite again (7) and obtain

$$\left|\frac{c_1}{t \cdot 6^{a-b} \pm t_1}\gamma^n 6^{-b} - 1\right| < \frac{1}{t \cdot 6^a \pm t_1 6^b} < \frac{1}{6^{a-1}} < \frac{36}{\gamma^{n-3}} < \frac{1}{\gamma^{n-16}}, \tag{18}$$

where we use $6^{a+1} > \gamma^{n-3}$ from (9). Let Λ_1 be the expression inside of the absolute value on the left side of (18). Again, being $\Lambda_1 \neq 0$ we consider

$$\alpha_1 = \frac{c_1}{t \cdot 6^{a-b} \pm t_1}, \alpha_2 = \gamma, \alpha_3 = 6, \quad b_1 = 1, b_2 = n, b_3 = -b.$$

and apply Matveev's inequality with $B = n$. We know the heights of α_2 and α_3 and for α_1, (17) gives

$$h(\alpha_1) < h(c_1) + h(t \cdot 6^{a-b} \pm t_1) < 5.8708 \times 10^{13}(1 + \log n).$$

So we take $A_1 = 1.76124 \times 10^{14}(1 + \log n)$ and A_2, A_3 as above. Then Matveev's inequality gives

$$\log|\Lambda_1| > -C \cdot (1 + \log n) \cdot 1.76124 \times 10^{14}(1 + \log n) \cdot 0.3 \cdot 5.4,$$

which compared with (18) yields

$$(n - 16)\log\gamma < 7.71631 \times 10^{26}(1 + \log n)^2.$$

Thus, $n < 1.09763 \times 10^{28}(\log n)^2$ and from Lemma 2.3 we get the following absolute upper bound on n:

$$n < 1.83028 \times 10^{32}. \tag{19}$$

Now we reduce this upper bound on n. Consider

$$\Gamma = n\log\gamma - a\log 6 + \log\frac{c_1}{t},$$

and go to (16). Assume that $a - b > 1$. Note that $e^\Gamma - 1 = \Lambda \neq 0$. Thus, $\Gamma \neq 0$ and with the same above analysis we find that

$$0 < |\Gamma| < \frac{2}{6^{a-b-1}}.$$

Dividing through by $\log 6$, we obtain

$$0 < |n\tau - a + \mu| < \frac{7}{6^{a-b}},$$

where

$$\tau := \frac{\log \gamma}{\log 6} \qquad \text{and} \qquad \mu_t := \frac{\log\left(\frac{c_1}{t}\right)}{\log 6} \quad \text{for} \quad t = 1, 2.$$

With $M = 1.83028 \times 10^{32}$ *Mathematica* finds that the 74-*th* convergent

$$\frac{p_{74}}{q_{74}} = \frac{119875645907448962613708293925797 9}{76382876539426574106903256420988 28}$$

of τ is such that $q_{74} > 6M$ and $\varepsilon_t = ||q_{74} \cdot \mu_t|| - M||q_{74} \cdot \tau|| > 0.107923 > 0$ for $t = 1, 2$. Thus from Lemma 2.2, with $A := 7$ and $B := 6$, we obtain that

$$a - b < \frac{\log(7 \cdot q_{74}/\varepsilon)}{\log 6} < 46.$$

Now consider

$$\Gamma_1 = n \log \gamma - b \log 6 + \log\left(\frac{c_1}{t \cdot 6^{a-b} \pm t_1}\right),$$

and go to (18). Then $\Gamma_1 \neq 0$ and we have

$$0 < |\Gamma_1| < \frac{2}{\gamma^{n-16}}.$$

Dividing through by $\log 6$, we obtain

$$0 < |n\tau - b + \mu| < \frac{101}{\gamma^n},$$

where τ is as above and

$$\mu := \frac{\log\left(c_1/(t \cdot 6^{a-b} \pm t_1)\right)}{\log 6}.$$

Let

$$\mu_{k,t,t_1} := \frac{\log\left(c_1/(t \cdot 6^k \pm t_1)\right)}{\log 6}, \quad \text{for} \quad k = 2, 3, \ldots, 45 \quad \text{and} \quad t \neq t_1 \in \{1, 2\}.$$

Again, *Mathematica* finds that the 74-*th* convergent of τ is such that $q_{74} > 6M$ and $\varepsilon_{k,t,t_1} \geqslant 0.00798086$ for all $k = 2, \ldots, 45$ and $t \neq t_1 \in \{1, 2\}$. Then the maximum of the $\log(q_{74} \cdot 101/\varepsilon_{k,t,t_1})/\log \gamma$ for all $k = 2, \ldots, 45$ and $t \neq t_1 \in \{1, 2\}$ is at most 311. So, $n \leqslant 311$ which contradicts the assumption on n and finish the proof of this case.

3.3. Case (6)

Again, $n \geqslant 3$, $n \neq 4$. Now $t \in \{1,3\}$ and $a \geqslant 0$. We have the inequalities

$$\gamma^{n-3} \leqslant P_n = t \cdot 6^a < 6^{a+1} \quad \text{and} \quad \gamma^{n-1} \geqslant P_n = t \cdot 6^a > 6^{a-1},$$

where $t \in \{1,3\}$. So, we use the same inequalities given in (9). Then $a \leqslant n$. In the interval $0 \leqslant n \leqslant 200$ and $0 \leqslant a \leqslant 32$ Mathematica gives the solution $P_6 = 3 \cdot 6^0 = 6^0 + 6^0 + 6^0$ listed in Theorem 1.1. We prove it is the only one in this case.

Let $n > 200$. Thus $a > 29$. From Binet's formula equation (6) gives

$$\left|\frac{c_1}{t}\gamma^n 6^{-a} - 1\right| < \frac{1}{\gamma^{n-10}}. \tag{20}$$

Let Λ be the expression inside of the absolute value on the left side of (20). As $\Lambda \neq 0$ we take

$$\alpha_1 = \frac{c_1}{t}, \alpha_2 = \gamma, \alpha_3 = 6, \quad b_1 = 1, b_2 = n, b_3 = -a.$$

and apply Matveev's inequality to it with $B = n$. The height of α_1 is

$$h(\alpha_1) < h(c_1) + h(t) < \log\gamma + 7\log 2.$$

So we take $A_1 = 15.4$ and $A_2 = 0.3, A_3 = 5.4$ as above. Hence, from Matveev's inequality we obtain

$$\log|\Lambda_1| > -C \cdot (1 + \log n) \cdot 15.4 \cdot 0.3 \cdot 5.4,$$

which compared with (20) and Lemma 2.3 yields

$$n < 3.24438 \times 10^{16}. \tag{21}$$

Now, consider

$$\Gamma = n\log\gamma - a\log 6 + \log\frac{c_1}{t},$$

and go to (20). Note that $e^\Gamma - 1 = \Lambda \neq 0$. Thus, $\Gamma \neq 0$ and we obtain in fact that

$$0 < |\Gamma| < \frac{2}{\gamma^{n-10}}.$$

Dividing through by $\log 6$, we obtain

$$0 < |n\tau - a + \mu| < \frac{19}{\gamma^n},$$

where

$$\tau := \frac{\log\gamma}{\log 6} \qquad \text{and} \qquad \mu_t := \frac{\log\left(\frac{c_1}{t}\right)}{\log 6} \quad \text{for} \quad t = 1, 3.$$

Now, with $M = 3.24438 \times 10^{16}$ *Mathematica* find that the 43-*th* convergent

$$\frac{p_{43}}{q_{43}} = \frac{53909443715906518}{343502498150492101}$$

of τ is such that $q_{43} > 6M$ and $\varepsilon_t = ||q_{43} \cdot \mu_t|| - M||q_{43} \cdot \tau|| > 0.087677 > 0$ for $t = 1, 3$. Thus from Lemma 2.2, with $A := 19$ and $B := \gamma$, we obtain that

$$n < \frac{\log(19 \cdot q_{43}/\varepsilon)}{\log \gamma} < 163,$$

which contradicts the assumption on n and finish the proof of this case. This finish the proof of Theorem 1.

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